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THE ZHUKOVSKII PROBLEM OF THE FLOW AROUND A SHEET PILE†

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The solution of the Zhukovskii problem of the flow around a sheet pile is given using the principles of two-dimensional steady-state seepage in the case when, accompanying the motion of the seeping water, there is a layer of saline ground waters at a certain depth under the sheet pile and this layer is located above an impermeable thickness of rock salt. The mixed boundary-value problem of the theory of analytic functions which arises is solved using Polubarinova-Kochina's method, which is based on the application of the analytical theory of linear differential equations and, also, the method, developed by us, of the conformal mappings of circular polygons in polar meshes, which are extremely typical for the velocity hodograph domains of such flows. While reflecting the specific details and individual properties of such flows, the solution constructed below turns out to be expressed in closed form in terms of elementary functions and, consequently, it is the simplest and most convenient solution. In addition, it is the most general solution for the class of problems being considered. The well known results Zhukovskii, Vedernikov and others are obtained from it as special and limiting cases. A detailed hydrodynamic analysis and the specific features of the seepage process being considered, as well as the effects of all the physical parameters of the model on the pattern of the phenomenon, are presented using this solution and by numerical calculations. © 1999 Elsevier Science Ltd. All rights reserved.

In his paper [1], published posthumously in 1923 (see also [2] in the treatment and with the remarks by Kochina), Zhukovskii modified Kirchoff's method in the theory of jet flows and extended it to solve seepage problems with free surfaces. Here, a special mathematical function (the Zhukovskii function) was introduced for the first time. This function has subsequently [3–5] found wide application in the theory of unconfined seepage. In particular, using it, Zhukovskii investigated the problem of the flow around a sheet pile (a Zhukovskii sheet pile) and constructed the solution using the character of the singularities of this function.

Zhukovskii's paper [1] opened up the possibility of the mathematical modelling of seepage problems with free boundaries and marked the beginning, in the view of many investigators, of the study of the above-mentioned class of seepage flows. For instance, Vedernikov [6], using the method of conformal mappings developed by him (which is known as the Vedernikov-Pavlovskii method [3, 4]) initially gave a different, more precise solution of the Zhukovskii problem and then, in [7–10], studied the effect of the ground capillarity on the seepage flow pattern. Subsequently [11], he obtained a solution of the problem for the case when a highly permeable layer, on the upper horizontal surface of which the pressure head has a constant value, underlies the seepage domain at a certain depth. This problem is treated in [12, 13] using the method of majorant domains for the case of a curvilinear surface of the drainage base. Other extensions of the Zhukovskii problem (when there is a vertical sheet, departing downwards to infinity, in the lower part of the flow, etc.) have been made in [4, 14–18].

Below we consider the case when there is a layer of stationary saline water accompanying the motion of the water under the Zhukovskii sheet pile. Polubarinova-Kochina's method [3] is used to solve this problem, as well as the method of conformal mappings of circular polygons in polar meshes [19, 20] that arise in the velocity hodograph domains of such flows. Using special schemes, the effect of the ground capillarity on the flow pattern is revealed. Special and limiting cases are noted: a scheme with a confining stratum, the flow around a sheet pile in a water-permeable stratum of unbounded capacity, taking the ground capillarity into account, Vedernikov's case (no evaporation) and the Zhukovskii problem (no evaporation or ground capillarity) [1, 3–6].

1. FORMULATION OF THE PROBLEM

The flow pattern is shown schematically in Fig. 1. The plane motion of fresh waters of density ρ_1 under a sheet pile *BCD*, which is impermeable to water, is considered when, at a certain depth, there

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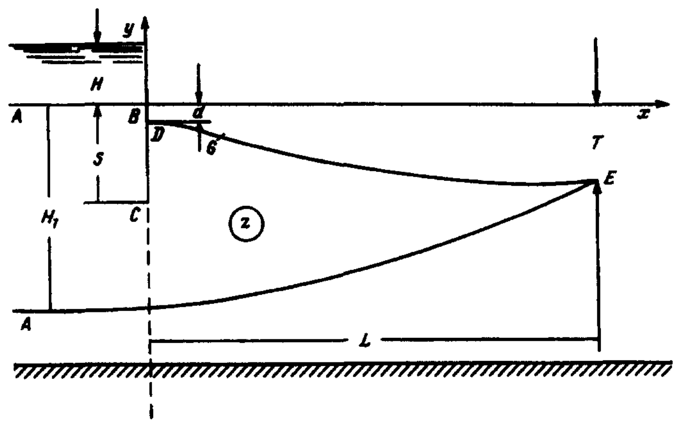

Fig. 1.

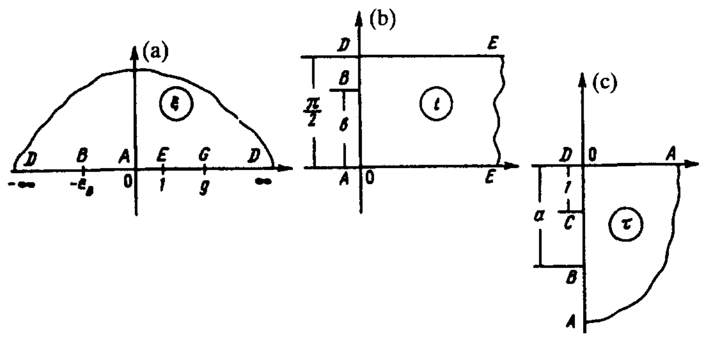

Fig. 2.

is a layer of salt water of density ρ_2 ($\rho_2 > \rho_1$) lying on a horizontal impermeable salt stratum. The fresh seepage waters, on flowing around the sheet pile, ascend behind it to a certain height *CD* and form a free surface *DE* from which there is evaporation at a constant intensity ε (relative to the seepage coefficient of the ground k) which compensates the seepage across the bottom of the water race (the canal or reservoir). At the same time, under the action of the pressure head H, the moving fresh waters displace the heavy saline waters (brine) such that the separation line *AE* between them starts to be deformed: it is depressed under the water race and rises to the right.

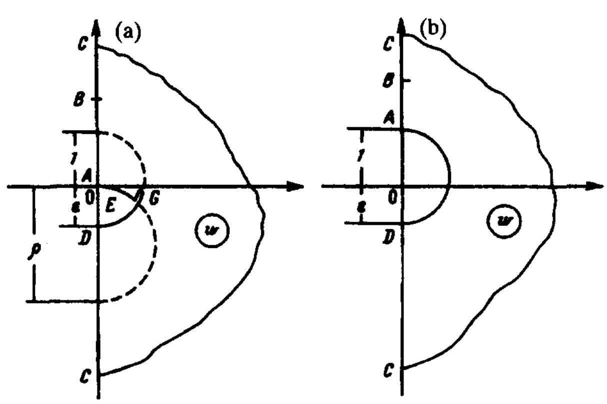

Fig. 3.

A steady-state motion is possible after a certain fairly long time when the brine is at rest and the separation line turns out to be a streamline for the fresh water [21, 22]. A distinctive formation of seeping ground waters then arises over the stationary salt waters—the so-called fresh-water lens. The determination of the lens boundaries enables one to establish the dimensions of the demineralization zone and, consequently, to estimate the possible stocks of fresh water.

The acting pressure head H, the sheet-pile length S and the depth T, from the ground surface, of the underlying ground waters outside the lens are taken as given. As is traditionally accepted in problems of this kind [3, 4], we neglect the capillary and diffusion effects on the boundary of the liquids.

We now introduce the complex flow potential $\omega = \varphi + i\psi$ and the complex coordinate $z = x + iy$, which are divided by kT and T, respectively. The problem consists of determining the depression curve DE and the separation line AE in the case of the following boundary conditions

$$\begin{aligned}&AB\colon y = 0,\ \varphi = -H;\quad BCD\colon\ x = 0,\ \psi = Q\\ &DE\colon\ \psi = Q-\varepsilon x,\quad \varphi = -(y + T);\quad AE\colon \psi = 0,\quad \varphi = \rho(y + H_1)-H\end{aligned} \tag{1.1}$$

Here H_1 is the squeezing depth to the left, for which we have ([3, p. 332]) the expression $H_1 = T + H/\rho$, $\rho = \rho_2/\rho_1-1$ and Q is the required seepage flow rate. Assuming $x = L$ in the first condition for the segment DE, we obtain

$$Q = \varepsilon L \tag{1.2}$$

This last condition expresses the fact that the flow rate is equal to the evaporation rate from the free surface under the steady-state seepage conditions. Hence, the magnitude of the flow rate is calculated using formula (1.2), after having determined the lens dimensions.

2. CONSTRUCTION OF THE SOLUTION

Polubarinova-Kochina's method [3] is used to solve the problem. This method is based on the use of the analytic theory of linear differential equations. We introduce the auxiliary variable ζ and the function $z(\zeta)$, which conformally maps the upper half-plane of ζ onto the z domain (the correspondence of the points is indicated in Fig. 2a), the complex velocity $w = d\omega/dz$ and also

$$Z = dz/d\zeta,\ F = d\omega/d\zeta \tag{2.1}$$

On determining the exponents of the functions Z and F near the singular points [3], we find that, in the given case, they are linear combinations of two branches of the following Riemann function [3, 23]

$$\begin{aligned}&P\begin{Bmatrix}-\zeta_B & 0 & 1 & g & \infty & \\ -\tfrac{1}{2} & -1 & -(1+\nu)/2 & 0 & \tfrac{3}{2} & \zeta\\ -\tfrac{1}{2} & -\tfrac{1}{2} & -(1-\nu)/2 & 2 & 2 & \end{Bmatrix} =\\ &= \frac{1}{\zeta\sqrt{(\zeta+\zeta_B)(1-\zeta)^{(1+\nu)}}}P\begin{Bmatrix}0 & 1 & g & \infty & \\ 0 & 0 & 0 & -\nu/2 & \zeta\\ \tfrac{1}{2} & \nu & 2 & -(1+\nu)/2 & \end{Bmatrix} =\\ &= \frac{Y}{\zeta\sqrt{(\zeta+\zeta_B)(1-\zeta)^{(1+\nu)}}},\quad \nu = \frac{2}{\pi}\operatorname{arc\,ctg}\sqrt{\varepsilon\frac{\rho+1}{\rho-\varepsilon}}\end{aligned} \tag{2.2}$$

Relation (2.2) reveals that $\zeta = \zeta_B$ is a regular point for the function Y. Therefore, the linear differential equation of the Fuchs class, which corresponds to the Riemann symbol (2.2), contains four regular singular points and takes the form

$$Y'' + \left(\frac{1}{2\zeta} + \frac{1-\nu}{\zeta-1} - \frac{1}{\zeta-g}\right)Y' + \frac{\nu(1+\nu)\zeta-\lambda}{4\zeta(\zeta-1)(\zeta-g)}Y = 0 \tag{2.3}$$

It is well known [3, 23, 24] that difficulties arise when integrating equations of this type, due to the fact that the coefficients of Eq. (2.3), apart from the undetermined affix g, contain an additional affix, the so called accessor parameter λ, which is also unknown in advance. These constants are not completely determined by the position of the singular points of the equation and the exponents in them and, up to now, no method exists for determining them. However, in the case under consideration, the form of Eq. (2.3) enables one to resort to direct integration in closed form in terms of elementary functions and to determine all the unknown constants in the equation.

We will now consider the complex velocity w domain, corresponding to boundary conditions (1.1), which is shown in Fig. 3(a). This domain, which is a cyclic quadrangle with two right angles, an arbitrary angle $\pi\nu$ at the vertex E and a cut DGE, belongs to the class of polygons in circular and polarnets [24–27], that is, bounded arcs of concentric circles and straight line segments which pass through the origin of the system of coordinates. Hence, it is convenient here to take the half-strip $0 < \mathrm{Re}\, t < \infty$, $0 < \mathrm{Im}\, t < \pi/2$ of the plane t (Fig. 2b) as the auxiliary domain of the parametric variable.

The change of variables

$$t = \mathrm{th}^2\zeta \tag{2.4}$$

maps the upper half-plane of ζ onto the half-strip of the plane t, and Eq. (2.3) is transformed in the following way

$$G(t)\mathrm{ch}^2 tY'' + \nu G(t)\mathrm{sh}\, 2tY' + [(\nu^2 + \nu - \lambda)\mathrm{sh}^2 t - \lambda]Y = 0 \tag{2.5}$$

$$G(t) = (g-1)\mathrm{sh}^2 t + g$$

According to the procedure for determining particular solutions of equations of this type [19, 20], Eq. (2.5) has two linearly independent integrals of the form

$$Y_k(t) = X_k(t)\mathrm{ch}^{-(1+\nu)} t, \quad k = 1.2$$

$$X_1(t) = \mathrm{ch}\, t\, \mathrm{ch}\, \nu t + C\, \mathrm{sh}\, t\, \mathrm{sh}\, \nu t, \quad X_2(t) = \mathrm{ch}\, t\, \mathrm{sh}\, \nu t + C\, \mathrm{sh}\, t\, \mathrm{ch}\, \nu t \tag{2.6}$$

Here C is a certain constant, to be determined, which controls the configuration of the cut and the position of its vertex G in the w plane. Note that, generally speaking, a cut is also possible along an arc of the circle AGE. When $C = 1$, the cut disappears and, in this case, the quadrangle degenerates into a triangle.

The function which accomplishes the conformal mapping of the half-strip of the t plane onto the specified cyclic quadrangle of the w plane must be expressed in terms of the ratio of linear combinations of the solutions Y_1 and Y_2. If such combinations are put together and use is made of the corresponding of the points A, E and D in the t- and w-planes, we obtain

$$w = \frac{\gamma\rho X_2(t)}{X_1(t) + i\gamma X_2(t)}, \quad \gamma = \sqrt{\frac{\varepsilon}{(\rho+1)(\rho-\varepsilon)}} \tag{2.7}$$

On considering relations (2.2) and (2.4) and taking account of (2.7), we find

$$Z = \frac{A}{\gamma\rho}\frac{X_1(t) + i\gamma X_2(t)}{\Delta(t)}, \quad F = A\frac{X_2(t)}{\Delta(t)}; \quad \Delta(t) = \mathrm{sh}\, t\sqrt{\mathrm{sh}^2 t + B^2}, \quad B = \sin b \tag{2.8}$$

where A is the modelling scale constant.

It can be verified that functions (2.1), which are defined on the basis of (2.8) and (2.4), satisfy boundary conditions (1.1), which have been formulated in terms of the above-mentioned functions, and they are therefore a parametric solution of the initial boundary-value problem.

3. SPECIAL AND LIMITING CASES

We will now mention some special and limiting cases of the initial seepage scheme which are associated with limiting values of the individual physical parameters.

A scheme with a confining stratum ($\rho = \infty$). We will first consider the limiting case $\rho = \infty(\rho_2 = 0)$, which can be interpreted as a "solidification" of the brine. When $\rho = \infty$, we have $\gamma = 0$, and expressions (2.7) and (2.8) are simplified.

When $\gamma = 0$, it is clear from the expression for Z that $\partial y/\partial\varphi = 0$ in AE and, consequently, $y = \mathrm{const}$, that is, we have an impermeable horizontal foundation.

A Zhukovskii sheet pile in ground of infinite depth ($T = \infty$). *Allowance for the ground capillarity.* Using the preceding model we will consider the case when the water-impermeable layer is located at a very great distance. It is obvious that this flow is obtained from the earlier scheme when the distance T increases without limit. In order to take the limit, we relocate the origin of the system of coordinates in the t plane to the point D and carry out the transformation $t = (\tau + hi)\pi/2h$, which transposes a half-strip in the t plane into a half-strip of height h in the τ plane. When $h \to \infty$, the points A and E approach one another, merging in the limit at infinity: the half-strip in the τ plane degenerates into the right lower square (Fig. 2c).

Now, on taking the limit as $T \to \infty$ and $h \to \infty$ in expressions (2.7) and (2.8) when $\gamma = 0$, we shall have

$$w = i\frac{\tau - i\varepsilon}{\tau + i}, \quad Z = A\frac{-i\tau + 1}{\sqrt{\tau^2 + a^2}}, \quad F = A\frac{\tau - i\varepsilon}{\sqrt{\tau^2 + a^2}} \tag{3.1}$$

The inflection point G in the z plane, on merging with the point D, reaches the y axis and becomes the point of maximum excess of the depression curve. Corresponding transformations were also performed with the complex velocity domain and, as a result of the merging of the points A and E at the point $w = i$, the semicircle $|w - (1 - \varepsilon)i/2| < (1 + \varepsilon)/2$ emerges from the domain w and takes the form of the lune shown in Fig. 3(b).

Integrating the last two equations of (3.1) and taking account of the ground capillarity, we find

$$\begin{gathered} z = -\chi\left(\sqrt{a^2 - \tau^2} - a + \arcsin\frac{\tau}{a}\right) - d \\ \omega = \chi\left(\sqrt{a^2 - \tau^2} - a - \varepsilon\arcsin\frac{\tau}{a}\right) + d + h_c \\ \chi = \frac{2(H + h_c)}{\pi(1 + \varepsilon)}, \quad a = \frac{\pi}{2} + \frac{d}{\chi}, \end{gathered} \tag{3.2}$$

where h_c is the height of the capillary rise of the water in the ground. We recall [3, p. 136; 7, p. 21] that the capillarity effect in such schemes appears in such a way as if, instead of a height of water in the water race H, there is a pressure head which is increased by an amount h_c, that is, it is equal to $H + h_c$.

As a result, the required dependences are obtained: the sheet-pile length

$$S = \sqrt{\left(\frac{\pi}{2}\chi + d\right)^2 - \chi^2} - \chi\arccos\frac{1}{\pi/2 + d/x} \tag{3.3}$$

and the equation of the free surface AE in the form of a catenary

$$y = \frac{\pi}{2}\chi - \left(\frac{\pi}{2}\chi + d\right)\operatorname{ch}\frac{x}{\chi} \tag{3.4}$$

Vedernikov's case ($\varepsilon = 0$). The solution of the problem of the flow around a Zhukovskii sheet pile when there is no evaporation is obtained from formulae (3.2)–(3.4) by putting $\varepsilon = 0$, and is identical with formulae (160) and (163) in [9, p. 112] and formulae (2.1) and (2.11) of [3, p. 135]. (In formulae (163) and (2.10) errors have apparently slipped in: in (163) the quantity H_v should appear instead of ih_v and the quantity $2(H + h_k)$ should appear in the second term of formula (2.10) instead of $H + h_k$ and the term $(H = h_k)$ is missing.)

The Zhukovskii problem ($\varepsilon = 0, h_c = 0$). The solution of the problem of the flow around a Zhukovskii sheet pile when there is no evaporation or ground capillarity follows from formulae (3.2)–(3.4) when $\varepsilon = 0, h_c = 0$ and is identical with the well-known results in [2, pp. 326–327].

4. CALCULATION OF THE FLOW SCHEME AND AN ANALYSIS OF THE NUMERICAL RESULTS

Writing out representations (2.8) for the different segments of the boundary of the domain t and integrating them, we obtain the parametric equations of the corresponding boundary segments of the model, which contain three unknown constants: A, C and B. The length of the channel S and the magnitude of the pressure head H serve to determine the mapping parameters C and B, and, here, the modelling constant A is eliminated in advance from all the equations by means of a relation which fixes the depth T of the saline ground waters outside the lens. The monotonicity of the functions occurring in these equations is revealed numerically and the unique solvability of the system for the unknown constants is thereby established.

After the required parameters have been found, the following quantities are to be determined: d (the length of the segment BD), L (the horizontal projection of the free surface DE), the flow rate Q from formula (1.2), the squeezing depth to the left of H_1. The coordinates of the points of the free surface DE and the separation line AE are also calculated.

Tables 1 and 2 show the results of calculations to reveal the effect of the quantities ε, ρ, S and H on the seepage characteristics d (negative values mean that the free surface rises above the abscissa), L and H_1. In each section of the tables, one of the parameters is varied while the remaining parameters are held at the values $H = 1.0$; $S = 0.5$; $T = 1.0$; $\varepsilon = 0.001$ and $\rho = 0.05$. In the case of this basic version, $H_1 = 21$; $d = 0.03786$; $L = 136.66$ and $Q = 0.13666$ were obtained. An analysis of the dependences of the required seepage characteristics H_1, d, L and Q on the governing physical parameters ε, ρ, S and H (when each of them is varied by a factor of 10 or more) leads to the following conclusions.

1. The lens increases in breadth when evaporation decreases: an increase in the width of the lens L by a factor of 15.2 is accompanied by a decrease in the parameter ε by a factor of 90. In this case, the quantities d and Q decrease by a factor of 8.5 and 5.9 respectively. The ordinate of point D of emergence of the seepage water from under the sheet pile is raised.

2. When the parameter ρ decreases and, consequently, the back pressure from the saline waters, drops there is now a substantial increase in the squeezing depth to the left H_1, that is, the lens mainly becomes deeper. For instance, when ρ is reduced by a factor of 10 (see the second section of Table 1), the depth H_1 increases also by a factor of almost 10. Here, on the background of an insignificant change in the magnitude of d (by only 4.2%), the lens width increases by a factor of 2.7.

3. As the sheet-pile length SS is increased, the lens width and, consequently, the flow rate, decrease to a small extent, by just 1–1.5%. In this case, the ordinate of point D drops down very significantly: the magnitude of d increases by a factor of 3.3.

4. The magnitude of the applied pressure head turns out to have the most significant effect on the dimensions of the lens. For example, when H is increased by a factor of 10, the width L and the depth H_1 increase by a factor of 10.1 (that is, in almost direct proportion to the change in H), and the elevation of the highest rise of the ground waters beyond the sheet pile increases sharply and rises above the abscissa: when $H = 2.0$, we have $d = -0.925$ (see the second section of Table 2). Note that the approximate equality $H + d \leqslant T$ (the lack of an exact value is solely due to the errors which arise in evaluating improper integrals) is satisfied in the case of the parameters given in this section. Furthermore, it can be seen that the changes in the magnitude of d here are of a non-monotonic nature: the parameter d reaches a maximum value close to zero when $H \leqslant 1$ (see the results for the basic version).

Table 1

$\varepsilon \times 10^4$	$d \times 10^4$	L	$\rho \times 10^2$	$d \times 10^4$	L	H_1
1	134	611.4	1	393	278.1	101
2	178	370.4	2	385	204.0	51
5	270	204.9	3	379	170.4	34
50	846	56.0	10	377	104.0	11
90	1134	40.1	∞	510	52.0	1

Table 2

S	$d \times 10^4$	L	H	$d \times 10^3$	L	H_1
0	−2	138.6	0.5	520	67.9	11
0.1	162	137.4	0.9	134	122.9	19
0.2	235	137.1	2	−925	274.4	41
1	541	136.0	3	−1890	412.2	61
2	774	134.9	5	−3925	688.0	101

5. The calculations show that, as the depth of the underlying saline ground waters T increases, the squeezing depth to the left increases and the ordinate of the point D drops down: this behaviour is quite natural from the physical point of view.

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